

# CONTACT STRUCTURES ON PRODUCTS

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**ABSTRACT.** By a strong symplectic fold on a closed manifold  $M$  we mean a decomposition  $M = W_- \cup_N W_+$ , where  $N$  is the common boundary of  $W_-$  and  $W_+$ , both parts are equipped with exact symplectic forms convex at the boundary  $N$  and given by the same contact form in a neighborhood of  $N$ . We show that if  $M$  has a strong symplectic fold and  $X$  is a closed contact manifold, then  $X \times M$  is contact under some additional mild conditions on  $X$ . Some new families of contact manifolds are obtained in this way.

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## 1. INTRODUCTION

An intricate question of contact topology is whether a closed almost contact manifold admits a contact structure. It is solved positively only in low dimensions : in dimension three by Martinet [Ma] and for simply connected closed manifolds of dimension 5 by Hansjörg Geiges [G1]. In higher dimensions it is still an open question. In the sequel we consider only cooriented contact structures, i.e. those given by contact forms.

In this paper we consider this problem for products  $M \times X$ , where  $X$  is contact and  $M$  is almost complex. Recently Geiges and Stipsicz [GS] gave a formula which yields a contact form on products  $M \times S^1$  for some closed  $M$ . Let us describe their construction in a slightly more general case than in [GS].

**Definition 1.1.** *A strong symplectic fold on a closed manifold  $M$  is a decomposition  $M = W_+ \cup_N W_-$ , where  $N = \partial W_+ = \partial W_- = W_+ \cap W_-$  together with exact symplectic forms  $\omega_+, \omega_-$  on respectively  $W_+, W_-$ , such that for a tubular neighborhood  $N \times [-1, 1]$  of  $N$  the forms satisfy the following convexity condition:*

- (1)  $\omega_- = d(e^t \lambda)$  on  $N \times [-1, 0] \subset W_-$  and  $\omega_+(t) = d(e^{-t} \lambda)$  on  $N \times [0, 1] \subset W_+$ , where  $t$  is the parameter of  $[-1, 1]$  and  $\lambda$  is a contact form on  $N$ .

An obvious example is the double  $W \cup (-W)$ , if  $W$  is a compact manifold with boundary which admits an exact symplectic form satisfying convexity condition 1.

In our terminology we follow Ana da Silva [dS]. She shows that on any closed stably almost complex manifold there exists a symplectic fold, i.e. a decomposition into two submanifolds as above and a 2-form which is symplectic except for the common boundary of the two parts, where the form has fold singularities. In our definition the forms  $\omega_{\pm}$  agree only after restriction to  $N$ , and they do not give any globally defined form on  $M$ .

**Theorem 1.2.** [GS] *If  $M^{2m}$  admits a strong symplectic fold, then  $M \times S^1$  is contact.*

**Proof.** Let  $d\phi$  denote the standard orientation form on  $S^1$  and  $p : M \times S^1 \rightarrow M$  be the projection. If  $\omega_{\pm} = d\alpha_{\pm}$ , then  $p^* \alpha_{\pm} + d\phi$  are contact forms outside  $N \times [-1, 1] \times S^1$ .

Choose smooth functions  $f, g : [-1, 1] \rightarrow \mathbb{R}$  such that:

- (1)  $g$  is odd, equal to 1 near  $t = -1$ , equal to  $-1$  near  $t = 1$ , and it is decreasing from  $-1$  to  $1$ ,
- (2)  $f$  is even, positive, equal to  $e^{\pm t}$  near  $\pm 1$  and increasing on  $[-1, 0]$ ,
- (3)  $f'g - g'f > 0$  on  $[-1; 1]$ .

Then the formula

$$\alpha = f\beta + g d\phi$$

on  $[-1, 1] \times N \times S^1$  yields a contact form on  $N \times [-1, 1] \times S^1$  (with contact form  $\beta$  on  $N$ ) which extends those defined above. In fact, it is not difficult to calculate:

$$\alpha \wedge (d\alpha)^n = n f^{n-1} (f'g - fg') dt \wedge \beta \wedge (d\beta)^n \wedge d\theta > 0.$$

□

Geiges and Stipsicz apply this formula to show, using [B], that every closed orientable 4-manifold  $M$  admits a strong symplectic fold structure, thus  $M \times S^1$  is contact in this case.

The main result of the present paper provides a construction of a contact form on the product  $M \times X$  if  $M$  admits a strong symplectic fold and  $X$  is a *symmetric* contact manifold (the precise definition of this notion is given in Section 2). The principal application is the case when  $M$  is the double of Stein manifold.

In the proof we follow the idea of [GS] together with the Giroux presentation of any closed contact manifold as an open book decomposition with exact symplectic page convex at the boundary. More precisely, we consider an explicit formula for a contact form in terms of the symplectic form on the page. Then we mix the two formulas to get a form on  $M \times X$ . The form obtained in this way is non-contact at some points. However, it is a confoliation and we show that it can be deformed, using the result of Altschuler and Wu [AW], to a form which is contact everywhere.

In general, the existence of a strong symplectic fold structure seems to be a difficult question. The following classical result of Eliashberg [E] (cf. also [W] and Ch. 6 of [G2]) is the basic tool to construct some examples. Let us recall that  $W$  is the *trace* of a (single) *surgery* of index  $k+1$  on  $M^{2n+1}$  if  $W$  is obtained by attaching a handle of index  $k+1$  to  $M \times [0, 1]$ . It means that  $W$  is diffeomorphic to  $M \times [0, 1] \cup_f (D^{k+1} \times D^{2n-k+1})$ , where  $f : S^k \times D^{2n-k+1} \rightarrow M \times \{1\}$  is the attaching map of the handle. In particular,  $\partial W = M \cup (-M')$ , where  $M' = (M - f(S^k \times D^{2n-k+1})) \cup (D^{k+1} \times S^{2n-k})$  is the result of the surgery on  $M$ . By *symplectization* of a contact form  $\lambda$  on  $M$  we mean the form  $d(e^t \lambda)$  on  $M \times \mathbb{R}$ .

**Theorem 1.3.** *Let  $\lambda$  be a contact form on a  $(2n+1)$ -dimensional manifold  $M$  and  $W$  is the trace of a surgery on  $M$  of index  $k+1$ , where  $1 \leq k \leq n$  and  $n > 1$ . If the almost complex structure on  $M \times [0, 1]$  determined by  $\lambda$  extends to  $W$ , then there exists an exact symplectic form  $\omega$  on  $W$  such that  $\omega$  is the symplectization of  $\lambda$  near  $M \times \{0\}$  and the symplectization of a contact form in a collar of  $M'$ . In particular,  $M'$  admits a contact form. Furthermore, if  $V$  is a compact connected almost complex  $(2n+2)$ -dimensional manifold ( $n > 1$ ) and  $V$  admits a Morse function maximal on  $\partial V$  such that indices of all critical points are less or equal to  $n+1$ , then  $V$  admits a symplectic structure such*

that  $\partial V$  is convex (the boundary is of contact type). A Morse function with the required properties exists if and only if  $V$  has the homotopy type of a CW-complex of dimension at most  $n + 1$ .

Let us call any manifold  $V$  having the above properties of *Stein type*. Thus the double of a manifold of Stein type admits a strong symplectic fold. Since any oriented closed surface has a strong symplectic fold structure, thus we get the previously known fact that the product of a contact (symmetric) manifold with a compact oriented surface is contact [AW, B, G1].

Beyond this, Theorem 1.3 together with our result gives the existence of contact forms on some more general product manifolds. In fact, let  $M$  be a strong symplectic fold. Then  $M \times S^k$  is contact if  $k$  is odd.

We get also stable existence of contact forms. By this we mean that for  $k \geq m$  there exists contact form on:

- (1)  $M^{2m} \times S^{2k+1}$  if  $M$  is closed and stably almost complex;
- (2)  $X^{2m-1} \times S^{2k+2}$  if  $X$  is closed almost contact.

## 2. MAIN THEOREM

Let us start with some technical preliminaries.

**Definition 2.1.** *An open book decomposition of  $X$  is given by*

- (1) *a codimension two submanifold  $B \subset X$  (binding),*
- (2) *a fibration  $\pi : X - B \rightarrow S^1$  with fibre  $P$  (page),*
- (3) *a tubular neighborhood  $U$  of  $B$  diffeomorphic to  $B \times D^2$*

*such that the monodromy of the fibration  $\pi$  is equal to the identity in  $P \cap U$  and  $\pi|_U$  can be identified with the standard projection  $B \times (D^2 - \{0\}) \rightarrow S^1$ .*

According to [Gi, GM], with any closed contact manifold  $X$  one can associate an open book decomposition with some additional structure on the page and on the binding, compatible with the given contact structure in the way we describe below (for further details see [G2]).

Each page  $P$  has a 1-form  $\beta'$  so that  $d\beta'$  is symplectic on  $P$ . Further, near the boundary  $\partial P \times [0, \epsilon]$  we have  $\beta' = e^{-t}\beta$  for some contact  $\beta$  on  $\partial P$ . Moreover, if  $f : P \rightarrow P$  is the monodromy of  $\pi$ , then  $f^*\beta' - \beta' = d\bar{\phi}$  for some function  $\bar{\phi} : P \rightarrow \mathbb{R}$ . From now on we assume that  $f^*\beta' = \beta'$ . Such contact manifolds will be called *symmetric*. Under this assumption the mapping  $F(x, \phi) = (f(x), \phi + 1)$  on  $P \times \mathbb{R}$  preserves the form  $\beta' + l \cdot d\phi$  for any  $l \in \mathbb{R}$  as the following easy computation shows:

$$(2.2) \quad F^*(\beta' + l \cdot d\phi) = F^*(\beta') + F^*(d\phi) = f^*\beta' + l \cdot d\phi = \beta' + l \cdot d\phi.$$

The form  $\beta' + l \cdot d\phi$  descends to  $(P \times \mathbb{R})/\sim \cong X - (B \times D^2)$  (and it is contact for  $l \neq 0$ .) As the monodromy  $f$  near the boundary  $\partial P$  is the identity, the form  $\beta' + l \cdot d\phi$  is equal to  $\beta e^r + l \cdot d\phi$  near the boundary  $B \times D^2$  in polar coordinates  $(r, \phi)$  on  $D^2$ . We easily now extend  $\beta' + d\phi$  to  $B \times D^2$  by the formula

$$\alpha = h_1(r)\beta + h_2(r)d\phi,$$

where

$$h_1(r) = \begin{cases} 2 & \text{for } r = 0 \\ e^{1-r} & \text{for } r \in [1; 2], \end{cases}$$

it is strictly decreasing with all derivatives at 0 vanish,

$$h_2(r) = \begin{cases} r^2 & \text{near } r = 0 \\ 1 & \text{for } r \in [1; 2] \end{cases}.$$

and  $h_1(r)h_2'(r) - h_1'(r)h_2(r) > 0$ . As another simple calculations shows, newly defined form is contact on  $X$ .

Recall now the main result of [AW]. On a closed manifold  $Y^{2m+1}$  consider a confoliation, i.e. a 1-form  $\alpha$  such that  $\alpha \wedge (d\alpha)^m \geq 0$ . The points  $x \in Y$  where  $\alpha \wedge (d\alpha)^m > 0$  are called contact (regular), the other (non-contact) points are called singular and the set of singular points will be denoted by  $\Sigma$ . Altschuler and Wu ([AW]) show that under some assumptions, the heat flow can deform the confoliation to a contact form. To describe those assumptions we choose a Riemannian metric  $g$  on  $Y$  and consider the form  $\tau = \star(\alpha \wedge (d\alpha)^{m-1})$ , where  $\star$  denotes the Hodge star operation. Then at every point  $x \in Y$  we define a subspace  $\mathcal{D} \subset TY_x$  as the orthogonal complement of  $Null(\tau)_p = \{V \in T_p Y : \iota_V \tau = 0\}$ . At a contact point the subspace  $\mathcal{D}$  has dimension  $2m$  and it is transversal to the Reeb vector of  $\tau$ . At a point where rank of  $d\alpha$  on  $\ker \alpha$  is  $2m - 2$ , the dimension of  $\mathcal{D}$  is 2, and  $\mathcal{D}$  is zero at points where rank of  $d\alpha|_{\ker \alpha}$  is less than  $2m - 2$ . A point  $x$  is called *accessible* if there is a smooth curve  $\sigma : [0, 1] \rightarrow Y$  such that  $\sigma'(t) \in \mathcal{D}$  and is non-zero for all  $t \in [0, 1]$ ,  $\sigma(0) = x$  and  $\sigma(1)$  is a contact point.

**Theorem 2.3.** [AW] *Suppose that  $Y$  is a closed manifold with a confoliation  $\alpha$ . If every non-contact point of  $Y$  is accessible, then  $Y$  supports a contact form  $C^\infty$ -close to  $\alpha$ .*

Our main theorem is the following.

**Theorem 2.4.** *If  $(X^{2m+1}, \alpha)$  is a symmetric closed contact manifold and  $M^{2n}$  admits a strong symplectic fold, then  $X \times M$  is contact.*

**Proof.** Consider the decomposition  $W_1 \cup (N \times [-1; 1]) \cup W_2$  and the forms  $\omega_+, \omega_-, \lambda$  given by the strong symplectic fold on  $M$ . Here  $N$  is the common boundary of  $W_+, W_-$ ,  $W_- = W_1 \cup (N \times [-1, 0])$ ,  $W_+ = (N \times [0, 1]) \cup W_2$ .

We can assume that the contact form  $\alpha$  is given by the Giroux - Mohsen construction using  $\beta, h_1, h_2$  as described above. Consider the 1-form

$$(2.5) \quad \tilde{\eta} = h_1(r)\beta + f(t)\lambda + h_2(r)g(t)d\phi$$

on  $B \times D^2 \times N \times [-1, 1]$ , where  $f, g$  are given in Theorem 1.2.

**Lemma 2.6.** *The form  $\tilde{\eta}$  extends to a form on  $X \times M$  which is contact in the complement of  $(B \times \{0\}) \times (N \times \{0\}) \subset X \times M$ .*

**Proof.** For every  $l \in \mathbb{R}$  the form  $\beta' + ld\phi$  is well-defined on  $X - B \times D^2$  as Formula 2.2 shows (for  $l \neq 0$  this form is contact). Therefore on  $(X - B \times D^2) \times N \times [-1; 1]$  we

can put  $\tilde{\eta} = \beta' + g(t)d\phi + f(t)\lambda$ . This definition extends Formula 2.5, and additionally we have

$$\tilde{\eta} \wedge (d\tilde{\eta})^{m+n} = (m+n-1)f^n(f'g - g'f)d\phi(d\beta')^m \wedge \lambda \wedge (d\lambda)^{n-1} \wedge dt > 0,$$

hence our formula still defines a contact form. Finally, on  $W_+ \times M$  and  $W_- \times M$  we set  $\tilde{\eta}$  to be equal to respectively  $\beta' + d\phi + \gamma_-$  and  $\beta' - d\phi + \gamma_+$ . These form extend our definition and both are clearly contact and compatible with prescribed orientation.

It remains to examine  $\tilde{\eta}$  on  $B \times D^2 \times N \times [-1; 1]$ . Direct computations give

$$\tilde{\eta} \wedge (d\tilde{\eta})^{m+n} = c_1(f'g(h_1h_2' - h_1'h_2) + fg'h_1'h_2)\beta \wedge d\beta^{m-1} \wedge dt \wedge \lambda \wedge d\lambda^{n-1} \wedge dr \wedge d\phi,$$

where  $c_1$  is a positive constant. Since  $h_1h_2' - h_1'h_2 > 0$ ,  $f'g \geq 0$ ,  $fg'h_1'h_2 \geq 0$ , we see that  $\tilde{\eta} \wedge (d\tilde{\eta})^{m+n} \geq 0$  and it vanishes if and only if  $f'g = 0$  and  $fg'h_1'h_2 = 0$ . The equality  $f'g = 0$  implies  $t = 0$ . Furthermore, for  $t = 0$  we have  $fg' > 0$ . To complete the proof notice that our assumptions on  $h_1, h_2$  yields  $h_1'h_2 = 0 \Leftrightarrow r = 0$ .  $\square$

We want to apply Theorem 2.3, so we need the accessibility condition to be satisfied. First we need to know that  $\text{rank } d\tilde{\eta} | \ker \tilde{\eta} = 2(m+n-1)$  on  $\Sigma = B \times \{0\} \times N \times \{0\}$ . Unfortunately,  $\text{rank } d\tilde{\eta} | \ker \tilde{\eta} < 2(m+n) - 2$  since  $d\tilde{\eta}|T(X \times M)|_\Sigma = 2d\beta + d\lambda$  and  $\tilde{\eta}|T(X \times M)|_\Sigma = 2\beta + \lambda$ . In order to remedy this we change the confoliation form making it asymmetric with respect to the decomposition  $W_1 \cup (N \times [-1; 1]) \cup W_2$ . Roughly speaking, we force in this way some more transversality along the singular set. Define the form  $\eta$  on  $X \times M$  by the formula

$$(2.7) \quad \eta = \begin{cases} e^{-1}(\beta' + d\phi + \gamma_-) & \text{on } B \times D^2 \times W_1 \\ k(t)(h_1(r)\beta + f(t)\lambda + h_2(r)g(t)d\phi) & \text{on } B \times D^2 \times N \times [-1; 1] \\ e(\beta' - d\phi + \gamma_+) & \text{on } B \times D^2 \times W_2. \end{cases}$$

Here  $(r, \phi)$  are polar coordinates on the disk  $D^2$  of radius 2,  $f, g$  are functions defined in Theorem 1.2 and  $k : M \rightarrow [e^{-1}; e]$  is a positive, non-decreasing function satisfying

$$k(t) = \begin{cases} e^{-1} & \text{on } W_1 \\ e^t & \text{on } N \times [-1 + \varepsilon; 1 - \varepsilon], \quad t \in [-1 + \varepsilon; 1 - \varepsilon] \\ e & \text{on } W_2 \end{cases}$$

and  $\varepsilon$  small enough.

This formula extends to  $X \times M$  as in Lemma 2.6. We get again a confoliation with the critical set  $\Sigma = B \times \{0\} \times N \times \{0\}$ .

To apply the result of [AW] we need a Riemannian metric  $\langle \cdot, \cdot \rangle$  on  $X \times M$ . Choose  $\langle \cdot, \cdot \rangle$  so that near  $\Sigma$  submanifolds  $N, I, B, D^2$  are pairwise orthogonal. We will check that  $\eta$  satisfies the assumption of Theorem 2.3. In fact, we show that for every point  $(b, v) \in B \times \{0\} \times N \times \{0\}$ , the radial path  $z(r) = (b, (r, \phi), 0, v) \subset B \times D^2 \times N \times I$  (with  $z'(r) = \frac{\partial}{\partial r} \in TD^2$  for  $r \in [0; 2]$  and any fixed  $\phi \in [0, 2\pi)$ ) satisfies  $z'(t) \in \mathcal{D}$ , hence every  $x \in \Sigma$  is accessible from a contact point. The proof is divided into two parts. First we show that on  $\Sigma$  we have  $\mathcal{D} = TD^2$  and then that  $z'(r) \in \mathcal{D}$  for  $r \in (0; 2]$ .

**Lemma 2.8.** *Under the assumptions above,  $\mathcal{D} = TD^2$  on  $\Sigma$ .*

**Proof.** By definition 2.7,  $\eta = e^t\tilde{\eta}$ , hence

$$(2.9) \quad d\eta = e^t d\tilde{\eta} + e^t d\tilde{\eta} =$$

$$= e^t dt(h_1(r)\beta + f(t)\lambda + h_2(r)g(t)d\phi) + \\ + e^t(h'_1(r)dr\beta + h_1(r)d\beta + f'(t)dt\lambda + f(t)d\lambda + h'_2(r)g(t)drd\phi + h_2(r)g'(t)dtd\phi).$$

Substituting  $t = r = 0$  in Formula 2.9 we get that  $\eta|T(X \times M)|_\Sigma = 2\beta + \lambda$  and  $d\eta|T(X \times M)|_\Sigma = 2d\beta + d\lambda + dt(2\beta + \lambda)$  where  $\Sigma = B \times \{0\} \times N \times \{0\}$ . As  $d\beta^m = 0, d\lambda^n = 0$ , we easily calculate:

$$\begin{aligned} \eta \wedge (d\eta)^{m+n-1} &= \eta \wedge (m+n-1)(2d\beta + d\lambda)^{m+n-2} \wedge dt \wedge (2\beta + \lambda) = \\ &= \eta \wedge (m+n-1)2^{m-1}(d\beta)^{m-1} \wedge (d\lambda)^{n-1} \wedge dt \wedge (2\beta + \lambda) = \\ &= C\beta \wedge (d\beta)^{m-1} \wedge \lambda \wedge (d\lambda)^{n-1} \wedge dt = C d\text{vol}_B \wedge d\text{vol}_N \wedge dt \end{aligned}$$

for some positive constant  $C$ . Thus  $\star(\eta \wedge (d\eta)^{m+n-1}) = \pm C d\text{vol}_{D^2}$  and  $\mathcal{D} = TD^2$ .  $\square$

It is unknown yet if we can extend our path  $z(r)$  beyond  $\Sigma$  since we do not know the behavior of  $\mathcal{D}$  on the complement of  $\Sigma$ , hence the second part of the proof concerns the case where  $r > 0$ . The proof is an elementary but long computation, hence we skip some parts of it.

As  $\eta$  is contact on  $X \times M - \Sigma$  we have that  $\mathcal{D}$  is  $2(m+n)$ -dimensional and the Reeb field of  $\tau$  is equal to  $\mathcal{D}^\perp$ . We will show that with respect to the previously chosen metric  $\langle \cdot, \cdot \rangle$  the Reeb field  $R_\tau$  on  $B \times (D^2 - \{0\}) \times N \times \{0\}$  is perpendicular to  $\frac{\partial}{\partial r}$ . Therefore once we show that for  $t = 0$  the Reeb field  $R_\tau$  is tangent to  $T = B \times S_r^1 \times N \times I$  (with  $S_r^1 = \{p \in D^2 : |p| = r\}$ ), or equivalently  $\tau$  is not non-degenerate on  $T$ , the proof is completed.

From now on we omit the wedge sign from the computations to make them more compact.

As in Lemma 2.8, substituting  $t = 0$  in Formula 2.9 gives  $\tilde{\eta}|T(X \times M)|_S = h_1(r)\beta + 2\lambda$  and  $d\tilde{\eta}|T(X \times M)|_S = h'_1 dr\beta + h_1 d\beta + d\lambda - h_2 dtd\phi$  on  $S = B \times D^2 \times N \times \{0\}$ . We obviously have  $(d\eta)^{m+n-1} = (dt\tilde{\eta} + d\tilde{\eta})^{m+n-1} = (d\tilde{\eta})^{m+n-1} + (m+n-1)(d\tilde{\eta})^{m+n-2} dt\tilde{\eta}$  on  $S$ . Further, as  $d\beta^m = 0, d\lambda^n = 0$  we get

$$\begin{aligned} (d\tilde{\eta})^{m+n-1} &= \binom{m+n-1}{n-1} (d\lambda)^{n-1} (h'_1 dr\beta + h_1 d\beta - h_2 dtd\phi)^m + \\ &\quad + \binom{m+n-1}{n-2} (d\lambda)^{n-2} (h'_1 dr\beta + h_1 d\beta - h_2 dtd\phi)^{m+1} = \\ &= (d\lambda)^{n-1} ((d\beta)^{m-1} (D_1 dr\beta + D_2 dtd\phi) + D_3 (d\beta)^{m-2} dr\beta dtd\phi) + D_4 (d\lambda)^{n-2} (d\beta)^{m-2} dr\beta dtd\phi \end{aligned}$$

for some functions  $D_i$  ( $i \in \{1, 2, 3, 4\}$ ) of variable  $r$ . In a similar manner we calculate  $dt\tilde{\eta}(d\tilde{\eta})^{m+n-2}$ :

$$\begin{aligned} dt\tilde{\eta}(d\tilde{\eta})^{m+n-2} &= dt\tilde{\eta}(h'_1 dr\beta + h_1 d\beta + 2d\lambda - h_2 dtd\phi)^{m+n-2} = dt\tilde{\eta}(h'_1 dr\beta + h_1 d\beta + 2d\lambda)^{m+n-2} \\ &= dt\tilde{\eta} \left( \binom{m+n-2}{n-1} (h_1 d\beta)^{m-1} (2d\lambda)^{n-1} + (m+n-2)(h_1 d\beta + 2d\lambda)^{m+n-3} h'_1 dr\beta \right). \end{aligned}$$

After arduous, but elementary computation we get that

$$\begin{aligned} \eta \wedge (d\eta)^{m+n-1} &= C_1 \beta (d\beta)^{m-1} dr\lambda (d\lambda)^{n-1} + C_2 \beta (d\beta)^{m-1} d\phi (d\lambda)^{n-1} dt + \\ &\quad + C_3 (d\beta)^{m-1} d\phi \lambda (d\lambda)^{n-1} dt + C_4 \beta (d\beta)^{m-2} dr d\phi \lambda (d\lambda)^{n-1} dt \\ &\quad + C_5 \beta (d\beta)^{m-1} dr d\phi \lambda (d\lambda)^{n-2} dt + C_6 \beta (d\beta)^{m-1} \lambda (d\lambda)^{n-1} dt \end{aligned}$$

for some functions  $C_i$ ,  $i = 1, \dots, 6$  of variable  $r$ . Furthermore,  $\beta' = \star(\beta(d\beta)^{m-2})$  in  $B$  and  $\lambda' = \star(\lambda(d\lambda)^{n-2})$  in  $N$  both have maximal ranks equal to respectively  $2m - 2$  and  $2n - 2$ . If we additionally set  $\beta_1 = \star((d\beta)^{m-2})$  in  $B$  and  $\lambda_1 = \star((d\lambda)^{n-2})$  in  $N$ , then

$$\tau = \star(\eta \wedge (d\eta)^{m+n-1}) = E_1 dtd\phi + E_2 \lambda_1 dr + E_3 dr\beta_1 + E_4 \beta' + E_5 \lambda' + E_6 drd\phi$$

again for some functions  $E_i$ ,  $i = 1, \dots, 6$  of variable  $r$ . The pullback of  $\tau$  to  $T$  via the inclusion  $T \hookrightarrow M \times M$  to  $B \times S_r^1 \times N \times \{0\}$  we get that

$$\star(\eta \wedge (d\eta)^{m+n-1}) = E_1 dtd\phi + E_4 \beta' + E_5 \lambda'.$$

Obviously the rank of this form is equal to  $2(m - 1) + 2(n - 1) + 2 < 2(m + n)$  which finishes the proof.  $\square$

### 3. APPLICATIONS

Let  $V$  be a compact oriented smooth manifold with boundary. The manifold  $D(V) = V \cup_{\partial V} (-V)$  obtained by gluing two copies of  $V$  along their common boundary (using the identity to identify the boundaries) we call the *double* of  $V$ .

As we noticed in Introduction,  $M = D(V)$  admits a strong symplectic fold if  $V$  is of Stein type. Then the product of  $M$  with any closed contact symmetric manifold is contact. The purpose of this section is to give some other examples. In fact, we will describe some operations leading to new examples.

Notice, however, that using merely doubles of manifolds of Stein type and the fact that odd-dimensional spheres are contact symmetric manifolds, we can construct some old and some new examples.

**Lemma 3.1.** *If  $k$  is odd, then  $S^k$  is a contact symmetric manifold.*

**Proof.** Sphere  $S^k$  has the following open book decomposition. For the page  $P$  we take disk  $D^{k-1}$  and we form the product  $D^{k-1} \times S^1$  and then we collapse each circle  $\{p\} \times S^1$  for all  $p \in \partial D^{k-1}$ . We obviously get sphere  $S^k$ . As the disk  $D^{k-1}$  has an exact symplectic form with Liouville field transverse to  $\partial D^{k-1}$ , our sphere is contact and symmetric (see [G2], Theorem 7.3.3).  $\square$

**Proposition 3.2.** *If  $X$  is a closed symmetric contact manifold, then the following manifolds admit contact structures:*

- (1)  $S^2 \times X$ ;
- (2)  $M \times X$ , if  $M$  admits a strong symplectic fold;
- (3)  $S^n \times S^{k_1} \times \dots \times S^{k_r}$ , if  $k_1 \geq k_2 + \dots + k_r$ ,  $n$  is odd and  $k_1 + \dots + k_r$  is even;
- (4)  $M \times X$ , if  $M$  is a closed orientable 4-manifold.

**Proof.** (1) and (2) are direct consequences of Theorem 2.4. For (3) it is enough to recall Theorem 1.3 and to notice that  $S^{k_1} \times \dots \times S^{k_r}$  is a double of a manifold of Stein type, hence it is a strong symplectic fold. To get (4) one has to use [B].  $\square$

We explain now some stable existence results. By that we mean the existence of contact structures on products of manifolds with a sphere of dimension large enough. This holds both for almost complex manifolds and for almost contact manifolds.

**Theorem 3.3.** *If  $M^{2m}$  is a stably almost complex closed manifold and  $k \geq m$ , then  $M \times S^{2k+1}$  is contact. If  $X^{2m-1}$  is a closed almost contact manifold and  $k \geq m$ , then  $X \times S^{2k+2}$  admits a contact form.*

**Proof.** Let us introduce first some notation. By a handle decomposition of a cobordism  $W^w$  between  $\partial_0 W$  and  $\partial_1 W$  we mean a decomposition  $W_0 = \partial_0 W \times [0, \varepsilon] \subset V_1 \subset V_2 \subset \dots \subset V_s = W$ , where  $V_i$  is obtained from  $V_{i-1}$ , for each  $i = 1, \dots, s$ , by attaching a handle  $h_i = D^{n_i} \times D^{w-n_i}$  (of index  $n_i$ ). Equivalently, we can assume that there exists a Morse function on  $W$  which is constant and minimal on  $\partial_0 W$ , constant and maximal on  $\partial_1 W$ , with indices of critical points  $n_1, \dots, n_s$ . We say that a cobordism is *elementary* of index  $r$  if it has a handle decomposition with only one handle, and the handle has index  $r$ .

If  $k \geq m$ , then  $M^{2m} \times S^{2k}$  is equal to the double of the manifold  $M \times D^{2k}$  of Stein type. Thus  $M \times S^{2k}$  carries a strong symplectic fold and by Theorem 1.2 we get a contact structure on  $M \times S^{2k} \times S^1$ . Consider the obvious cobordism from  $M \times S^{2k} \times S^1$  to  $M \times S^{2k+1}$  given by surgery on  $D^{2m+2k} \times S^1$ . Since the space of complex structures (compatible with a fixed orientation) on  $\mathbb{R}^{2m+2k}$  is simply connected, the stable almost complex structure on  $M \times S^{2k}$  extends to the cobordism.

For the further arguments we have to explain the connection between handle decomposition of a manifold and that of its product with an elementary cobordism.

**Lemma 3.4.** *If  $K$  is a closed manifold of dimension  $k$ ,  $W^w$  is an elementary cobordism of index  $r$ , then there exists a handle decomposition of the cobordism  $K \times W$  with indices of all handles not exceeding  $k + r$ .*

**Proof of Lemma 3.4.** Let  $h = D^r \times D^{w-r}$  be the unique handle of  $W$ , so that  $W$  is obtained by attaching  $h$  to  $\partial_0 W \times [0, \varepsilon]$ . Let  $h_1, h_2, \dots, h_s$  be handles of a decomposition of  $K$  attached consecutively starting with handles of index 0. If  $h_1 = D^k$  is a handle of index 0, then we attach  $h_1 \times h = D^k \times D^r \times D^{w-r}$  to  $K \times \partial_0 W \times [0, \varepsilon]$  using the embedding  $D^k \times S^{r-1} \times D^{w-r} \rightarrow K \times \partial_0 W \times \{\varepsilon\}$  given by the attaching map of  $h$ . We then continue adding product handles  $h_i \times h = D^{n_i} \times D^r \times D^{k-n_i} \times D^{w-r}$ ,  $i = 2, \dots, s$  (here  $n_i$  denotes the index of  $h_i$ ). The attaching map of  $h_i \times h$  is defined on  $\partial(D^{n_i} \times D^r) \times D^{k-n_i} \times D^{w-r} \cong (S^{n_i-1} \times D^{k-n_i} \times D^r \times D^{w-r}) \cup D^{n_i} \times D^{k-n_i} \times S^{r-1} \times D^{w-r}$  by the attaching maps of  $h_i$  and  $h$ . In this way we get a handle decomposition of  $K \times W$ . Since the handle  $h_i \times h$  has index  $n_i + r \leq k + r$ , the lemma follows.  $\square$

We apply this lemma to finish the first part of Theorem 3.3. From the elementary cobordism of index 2 between  $S^{2k+1} \times S^1$  and  $S^{2k+2}$  we get an almost complex cobordism between  $M \times S^{2k} \times S^1$  and  $M \times S^{2k+1}$  with handles of indices not exceeding the half of dimension. By contact surgery 1.3, this gives a contact form on the convex end  $M \times S^{2k+1}$ .

Similar arguments yield the second statement. If  $X$  is almost contact, then  $X \times S^1$  is almost complex. Thus  $X \times S^1 \times S^{2k+1}$  is contact by the previous part. As above we can perform contact surgeries and get a symplectic cobordism yielding a contact form on  $X \times S^{2k+2}$ .  $\square$

We will describe now an example of a modification which can be performed on a manifold with a strong symplectic fold. Assume that  $M^{2m}$  admits a strong symplectic fold  $W_- \cup W_+$  with the fold locus  $N$ . We say that a surgery on a sphere  $S^{k-1} \subset M$  is



symmetric of index  $k$ , if it is performed using an embedding  $\phi : S^{k-1} \times D^{2m-k+1} \rightarrow M$  such that  $\phi = \phi_0 \times id_{D^1}$ , where  $\phi_0 : S^{k-1} \times D^{2m-k} \rightarrow N$  is an embedding and  $D^1$  corresponds to the transversal disk of a tubular neighborhood of  $N$ .

**Proposition 3.5.** *If  $M'$  is obtained from  $M$  by a symmetric surgery of index  $k \leq m = \frac{1}{2} \dim M$  such that the stable almost complex structure of  $M$  extends to  $M'$ , then  $M'$  has a strong symplectic fold structure.*

**Proof.**  $M' = (M - \phi(S^{k-1} \times D^{2m-k+1})) \cup (D^k \times S^{2m-k})$ . Decompose  $S^{2m-k}$  into the sum of two disks  $D_- \cup D_+$  such that decomposition corresponds to cut of the sphere by  $N$ . Since the surgery is symmetric, we get accordingly handles  $D^k \times D_-$ ,  $D^k \times D_+$  of index  $k$  attached to respectively  $W_-$ ,  $W_+$ , resulting in a decomposition  $W'_- \cup W'_+$ . Since  $k \leq m$  and almost complex structures on  $W_-$ ,  $W_+$  extend to these handles, we can also extend the given contact forms on  $W_\pm$ .  $\square$

**Corollary 3.6.** *If  $M^{2m}$  admits a strong symplectic fold,  $k + n = 2m$ , then so does the connected sum  $M \# (S^k \times S^n)$ .*

**Proof.** The previous proposition can be applied, since the connected sum is obtained by the surgery on a trivially embedded sphere  $S^{k-1}$ , and we can assume that  $k \leq n$ .  $\square$

Finally, we should admit that we do not know any example of closed stably almost complex manifold which admits no strong symplectic folds. Standard Morse - Smale theory shows that for any closed manifold  $M^{2m}$  one can find a decomposition  $M = W_+ \cup_N W_-$ , where  $N = \partial W_+ = \partial W_- = W_+ \cap W_-$  with both  $W_+$ ,  $W_-$  having the homotopy type of complexes of dimension at most  $m$ . If  $M$  is stably almost complex, then  $W_\pm$  are almost complex, thus we have exact symplectic forms on both parts by contact surgery. However, the resulting contact forms  $\lambda_-$ ,  $\lambda_+$  on  $N$  do not need to agree. What we do know, it is that they define homotopic almost contact structures on  $N$ .

Given two contact forms  $\mu_-$ ,  $\mu_+$  on  $N$ , let us say that  $\mu_+$  *dominates*  $\mu_-$  if there is a symplectic form on  $N \times [0, 1]$ , equal to  $\mu_-$  and concave at  $N \times \{0\}$ , equal to  $\mu_+$  and convex at  $N \times \{1\}$ . In this terminology the remaining question is whether there is an obstruction to find a contact form  $\lambda$  on  $N$  which dominates both  $\lambda_-$  and  $\lambda_+$ . For symplectic cobordism, non-zero algebraic torsion [Gi, LW] in contact homology is an obstruction of such kind, but it bans only in some cases symplectic cobordisms from a contact structure to another one, not from two structures to another one which dominates both.

#### 4. APPENDIX: COMPUTATIONS

We present here some of the calculations which led us to the proof of 2.4. The result was first checked using Mathematica's package "Differential forms" by Frank Zizza and Ulrich Jentschura [FZ] in low dimensions. Namely, for  $t_1 = 0$  (for technical reasons we slightly change notation to adapt it for our purposes) and around a point  $(b, d, n, 0) \in B \times D^2 \times N \times I$  we take coordinate system in which  $\beta = d[z_1] + x_1 d[y_1]$ ,  $\lambda = d[z_2] + x_2 d[y_2]$ . Further, on disk  $D^2$  we take coordinate system  $(x, y)$ . In this system we set  $h_1 = 2 - (x^2 + y^2)^2$  and  $h_2 = x^2 + y^2$  (hence in the formula below  $h_1$  is equal to  $2 - r^4$  near  $r = 0$  so that it is of class  $C^3$ ). Then the following expressions are equal respectively to  $\eta$  and  $d\eta$  :

$$\text{eta1} := (2 - (x^2 + y^2)^2)(d[z1] + x1d[y1]) + (d[z2] + x2d[y2])$$

$$\begin{aligned} \text{deta1} := & (-4x^3 - 4xy^2)d[x] \wedge d[z1] + (-4x^3x1 - 4xx1y^2)d[x] \wedge d[y1] + \\ & (-4x^2y - 4y^3)d[y] \wedge d[z1] + (-4x^2x1y - 4x1y^3)x1d[y] \wedge d[y1] + \\ & (2 - x^4 - 2x^2y^2 - y^4)d[t1] \wedge d[z1] + (2 - x^4 - 2x^2y^2 - y^4)d[x1] \wedge d[y1] + \\ & (2x1 - x^4x1 - 2x^2x1y^2 - x1y^4)d[t1] \wedge d[y1] + d[x2] \wedge d[y2] + \\ & d[t1] \wedge d[z2] + x2d[t1] \wedge d[y2] - xd[t1] \wedge d[y] + yd[t1] \wedge d[x] \end{aligned}$$

Now  $\tau = \star(\eta \wedge (d\eta)^3)$  can be computed in two steps: first we calculate

$$\text{ExteriorProduct}[\text{eta1}, \text{deta1}, \text{deta1}, \text{deta1}]$$

and later

$$\text{HodgeStar}[\%, t[x1, x1] + t[y1, y1] + t[z1, z1] + t[x2, x2] + t[y2, y2] + t[z2, z2] + t[x, x] + t[y, y] + t[t1, t1]]$$

where the percent sign refers to  $\eta \wedge (d\eta)^3$ .

Then  $\tau = \star(\eta \wedge (d\eta)^3)$  is given by

$$\begin{aligned} & (24(x^2 + y^2)^2)dx1 \wedge dy1 + (-96x(-1 + x1)x1y(x^2 + y^2)^2)dt1 \wedge dx1 + \\ & (-24x1(x^2 + y^2)(x^2 + x1y^2))dx1 \wedge dz1 + (6x(-2 + x^4 + 2x^2y^2 + y^4))dx \wedge dz1 \\ & + (6y(-2 + x^4 + 2x^2y^2 + y^4))dy \wedge dz1 + (-24(x^2 + y^2)^2(-2 + x^4 + 2x^2y^2 + y^4))dx2 \wedge dy2 + \\ & (24x2(x^2 + y^2)^2(-2 + x^4 + 2x^2y^2 + y^4))dx2 \wedge dz2 + (6x(-2 + x^4 + 2x^2y^2 + y^4)^2)dx \wedge dz2 + \\ & (6y(-2 + x^4 + 2x^2y^2 + y^4)^2)dy \wedge dz2 + (24x(x^2 + y^2)(-2 + x^4 + 2x^2y^2 + y^4))dt1 \wedge dy + \\ & (-24y(x^2 + y^2)(-2 + x^4 + 2x^2y^2 + y^4))dt1 \wedge dx \\ & + (-24(-1 + x1)x1y^2(x^2 + y^2)(-2 + x^4 + 2x^2y^2 + y^4))dx1 \wedge dz2 \end{aligned}$$

and  $\tau^4$  is equal to

$$\begin{aligned} & (-1990656(x^2 + y^2)^6(-2 + x^4 + 2x^2y^2 + y^4)^3)dt1 \wedge dx \wedge dx1 \wedge dx2 \wedge dy \wedge dy1 \wedge dy2 \wedge dz1 + \\ & (-1990656x2(x^2 + y^2)^6(-2 + x^4 + 2x^2y^2 + y^4)^3)dt1 \wedge dx \wedge dx1 \wedge dx2 \wedge dy \wedge dy1 \wedge dz1 \wedge dz2 + \\ & (-1990656x1(x^2 + y^2)^6(-2 + x^4 + 2x^2y^2 + y^4)^4)dt1 \wedge dx \wedge dx1 \wedge dx2 \wedge dy \wedge dy2 \wedge dz1 \wedge dz2 + \\ & (-1990656(x^2 + y^2)^6(-2 + x^4 + 2x^2y^2 + y^4)^4)dt1 \wedge dx \wedge dx1 \wedge dx2 \wedge dy \wedge dy1 \wedge dy2 \wedge dz2, \end{aligned}$$

As  $\iota_R d\text{vol}_{\mathbb{R}^9} = \tau^4$  for some  $R \in \text{lin}\{\frac{\partial}{\partial z1}, \frac{\partial}{\partial z2}, \frac{\partial}{\partial y1}, \frac{\partial}{\partial y2}\}$ , hence the Reeb field  $R_\tau$  of  $\tau$  is equal to  $R$  because  $\iota_R \tau^4 = \iota_R \iota_R d\text{vol}_{\mathbb{R}^9} = 0$ . The field  $R_\tau$  is obviously perpendicular to  $\frac{\partial}{\partial r}$  (away from the degenerate set  $\Sigma$ ).

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